Method of Synthesizing Nonuniform Waveguides

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Abstract—**A method is proposed for the synthesis of continuous nonuniform waveguides with rectangular cross section so that they show desired electromagnetic properties for discrete frequencies when excited by the TE_{10} mode. Starting from a uniform structure with known properties, the shape of the nonuniform waveguide is attained step by step by small systematic deformations.**

To show the feasibility of the method proposed, the mathematical formalism and numerical results are presented for reactive one-ports and filters with simple properties. In these cases, the problem is reduced to the solution of an equivalent resonator problem, i.e., a nonuniform waveguide resonator is developed for which a certain set of resonance modes occur at desired frequencies.

I. Introduction

A. General Aspects

NONUNIFORM waveguides can be used for the solution of transfer problems in microwave circuits. In this paper, the case of nonuniform waveguide is considered, where the cross section varies continuously along the waveguide. Compared to a nonuniform waveguide of equal length but only discrete discontinuities, the continuous configuration is superior with regard to bandwidth, loss, and high-power throughput.

As of this date, no method of synthesizing this type of nonuniform waveguide (in contrast to nonuniform TEM transmission lines) has been described in literature available to the author. The topic could therefore be of more general interest. This paper presents a short summary of [1]. It is, of course, impossible to take all the detailed problems solved in the above-mentioned reference into consideration. It should be pointed out that, in the meantime, further applications of the general synthesizing principle have arisen which cannot, however, be dealt with in this paper.

B. The General Synthesizing Principle

The method of solution is an iterative procedure that may be described in the following manner: the starting point is a waveguide which is uniform. In this case it is simple to compute its electromagnetic properties. The nonuniform waveguide with the desired properties is now approached step by step by small systematic deformations of the walls of the waveguide. This yields a linear relationship between the change of the structure and the change of its properties for each incremental step ($i - 1 \rightarrow i$). The mathematical formalism necessary for each step can be obtained by first-order perturbation theory. On the other hand, it is possible to use immediately well-known expressions for resonance frequency or input impedance, power consumption or other physical quantities, expressions which normally are used to determine the properties of waveguide structures by means of the variational method.

Prior to going into mathematical detail, it is necessary to make some basic assumptions.

C. Basic Assumptions

1) Only one kind of nonuniform waveguide is considered. It is of rectangular cross section and is completely described by a "characteristic longitudinal section" F and the width A as shown in Fig. 1(a).

2) The frequency band and the dimensions of the cross sections Q_1, Q_2 at either end of the waveguide (the cross sections of the uniform waveguide junctions) are chosen such that only the TE_{10} mode [Fig. 1(b) and (c)] can excite the structure. In this case, the components of the electromagnetic field versus the coordinate w are proportional to $\cos((\pi/A)\ w)$, $\sin((\pi/A)\ w)$ [Fig. 1(b)] or else are constant [Fig. 1(c)].

3) The frequency band, the dimensions of the cross sections Q_1, Q_2, and the shape of the physical transitions are chosen such that higher modes (TE_{11}, TM_{11}, $\cdots$ TE_{1n}, TM_{1n}, and TE_{20} $\cdots$ $TE_{m,0}$, respectively) excited inside the nonuniform structure can be neglected on account of their aperiodic attenuation at Q_1, Q_2. Thus the nonuniform structure is equivalent to a linear two-port. If there is a shortcut in the cross section Q_2, the structure is equivalent to a linear one-port.

4) Losses in the nonuniform waveguide and the medium are neglected. The medium is taken to be isotropic with a dielectric constant ϵ_0 and a permeability constant μ_0.

D. Basic Field Relationships

In the past, nonuniform waveguides of this type have been analyzed in detail [2]–[4]. By the selection of the

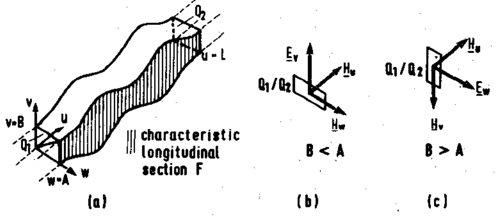

Fig. 1. Fundamental structure of the nonuniform waveguide to be synthesized and admissible excitations.

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particular fundamental structure and its excitation it is possible to reduce the three-dimensional problem to a two-dimensional problem by separation of the coordinate w in Fig. 1(a). The domain of definition of the reduced wave equation is now the longitudinal section F. In order to simplify the formulation of the boundary conditions, a suitable curvilinear coordinate system is introduced in F. This is performed by conformal mapping of the rectangular F' in the z plane to the longitudinal section F in the w plane as shown in Fig. 2. In principle, the computation of the complete electromagnetic field is now possible after solving a general wave equation for a scalar function $\phi(x,y)$:

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} + \gamma f(x,y)\phi(x,y) = 0 \qquad (1)$$

taking into account the simplified boundary conditions at the limits of the domain of definition which is now F or F'.

In (1),

$$\gamma = \omega^2\epsilon_0\mu_0 - \left(\frac{\pi}{A}\right)^2 \qquad (2)$$

for an excitation as shown in Fig. 1(b), and

$$\gamma = \omega^2\epsilon_0\mu_0 \qquad (3)$$

for an excitation as shown in Fig. 1(c).

The function $f(x,y)$ describes the nonuniform structure completely. Mathematically, $f(x,y)$ is the deformation of the infinitesimal area $d\sigma'$ when transferred to $d\sigma$ (Fig. 2):

$$d\sigma = f(x,y)\, d\sigma'. \qquad (4)$$

$f(x,y)$ must satisfy the equation [1]

$$\frac{\partial^2[\ln(f(x,y))]}{\partial x^2} + \frac{\partial^2[\ln(f(x,y))]}{\partial y^2} = 0 \qquad (5)$$

because the real and imaginary components of the corresponding conformal mapping function have to satisfy the differential equations of Cauchy and Riemann.

If a uniform rectangular waveguide (longitudinal section F', width A) is filled with a nonuniform medium which, in case of an excitation corresponding to Fig. 1(b), also is an anisotropic medium, the same type of general wave equation is obtained. Therefore, a physical interpretation may also be given for $f(x,y)$ [2]–[4].

After the general introduction to the fundamental equations of the nonuniform waveguides, the mathematical formalism for the process of synthesis itself must be considered. In order to demonstrate the feasibility of the method, only nonuniform waveguides with a relatively simple behavior are treated in this paper, such as reactive one-ports or bandpasses [1].

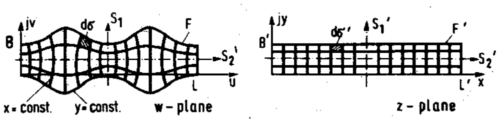

Fig. 2. Introduction of a curvilinear coordinate system in the characteristic longitudinal section F by conformal mapping.

II. Feasibility of the Method for Structures with Simple Electromagnetic Behavior

As shown in Section III, the problem is reduced to the solution of an equivalent resonator problem, i.e., a nonuniform waveguide resonator is developed for which a certain set of resonance modes occur at desired frequencies or eigenvalues $\{\gamma_p\}$. Ideal short circuits or open circuits at both ends of the nonuniform waveguide structure form a resonator. Now the problem is to synthesize such resonators. The starting point is a uniform waveguide with a known resonance mode spectrum. The TE_{10p} modes are of interest. The corresponding eigenvalues must be displaced step by step towards the desired values. It is not necessary to influence the higher order modes in the same way because their effect on the behavior of the nonuniform waveguide can be neglected in the considered frequency band if the following conditions are satisfied.

1) A series expansion of the corresponding fields in the cross sections Q_1, Q_2 yields only small TE_{10} components.

2) The resonance frequencies corresponding to the modes developed from the TE_{111}, TM_{111}, and TE_{201} modes, respectively, are outside the considered frequency band.

The process of synthesis is only possible by means of a computer because ten steps or more are required to get to the desired nonuniform structure. A simplified flow chart of the computer program is shown in Fig. 3.

A. *Mathematical Formalism of the Process of Synthesis*

Regarding (1) and the corresponding boundary conditions which are now only of a Dirichlet or Neumann type, a solution for an inverse eigenvalue problem must be found which is the function $f(x,y)$ derived from a known set of eigenvalues $\{\gamma_1,\gamma_2 \cdots \gamma_{pm}\}$. To get to a linear relationship between the incremental correction of the eigenvalues and the changes of the longitudinal section F_{i-1} for each step i, it is necessary to consider neighboring statuses $i-1$, i for the functions which describe the nonuniformities, the sets of corresponding eigenvalues, and the systems of corresponding eigenfunctions:

$$f_{i-1}(x,y) \rightarrow f_i(x,y)$$
$$\{\gamma_{i-1,p}\} \rightarrow \{\gamma_{i,p}\}$$
$$\{\phi_{i-1,p}\} \rightarrow \{\phi_{i,p}\}, \qquad i = 1,2 \cdots i_m, \qquad p = 1,2 \cdots p_m. \qquad (6)$$

From the general wave equation (1) the following expression can be derived for each function $f_i(x,y)$:

$$\gamma = \frac{\int_{F'} \phi(x,y)\left(-\frac{\partial^2\phi(x,y)}{\partial x^2} - \frac{\partial^2\phi(x,y)}{\partial y^2}\right) d\sigma'}{\int_{F'} f_i(x,y)\phi^2(x,y)\, d\sigma'}, \qquad i = 1,2 \cdots i_m. \qquad (7)$$

It is well known that γ is a stationary value $\gamma = \gamma_{i,p}$ whenever $\phi(x,y)$ is an eigenfunction $\phi_{i,p}(x,y)$. Therefore, if the eigenfunction $\phi_{i-1,p}(x,y)$ is used as a trial function,

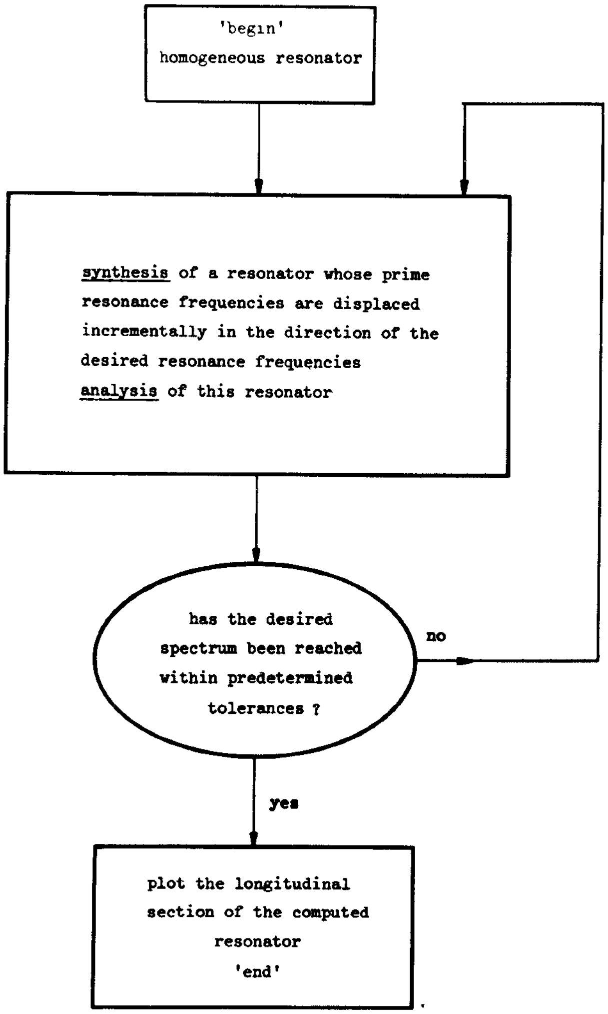

Fig. 3. Simplified flow chart of the computer program.

$\gamma_{i,p}$ is given with an error of second order:

$$\gamma_{i,p} \approx \frac{\int_{F'} \phi_{i-1,p}(x,y)\left(-\frac{\partial^2\phi_{i-1,p}(x,y)}{\partial x^2} - \frac{\partial^2\phi_{i-1,p}(x,y)}{\partial y^2}\right) d\sigma'}{\int_{F'} f_i(x,y)\phi_{i-1,p}{}^2(x,y)\, d\sigma'}, \qquad i = 1,2\cdots i_m, \qquad p = 1,2\cdots p_m. \tag{8}$$

If the function $f_i(x,y)$ were known (case of analysis), (8) yields a very good approximation for the exact eigenvalue. Vice versa, if a set of eigenvalues is required (case of synthesis), (8) can be considered as a functional equation for the computation of the unknown function $f_i(x,y)$.

To get a system of linear equations, a suitable series expansion for $f_i(x,y)$ is needed with a definite number of free parameters or coefficients. First, this series expansion must satisfy (5). Further, it must result in a smooth transition to the cross section of the homogeneous waveguide junctions at the ends of the nonuniform waveguide. Finally, a number s of free parameters must be reserved, to get, for instance, a resonator with a certain length L and a certain height B at the ends or at half-length.

For the computation of $f_i(x,y)$ an expression

$$f_i(x,y) = f_{i-1}(x,y)\left(1 + \sum_{p=0}^{p_m+s-1} \delta f_{i,p}\theta_p(x,y)\right) \tag{9}$$

where

$$f_0(x,y) \equiv 1$$

$$\theta_p(x,y) = \cosh\left[\frac{2p\pi}{L'}\left(y - \frac{B'}{2}\right)\right]\cos\left(\frac{2p\pi}{L'}x\right)$$

$$i = 1,2\cdots i_m.$$

is typical. The functions $\theta_p(x,y)$ satisfy the potential equation.

For the computation of the unknown coefficients $\delta f_{i,p}$, both (8) and conditions of the form

$$r_{i,t} = \int_{z_{0,t}}^{z_{1,t}} [f_i(x,y)]^{1/2}\,|dz| \tag{10}$$

with

$$i = 1,2\cdots i_m \qquad t = 1,\cdots s$$

$$r_{i,1} = B_i \qquad r_{i_m,1} = B \qquad r_{i,2} = L_i \qquad r_{i_m,2} = L\cdots$$

must be satisfied. In order to make sure that $f_{i_m}(x,y)$ satisfies (5), (9) becomes

$$f_i(x,y) = f_{i-1}(x,y)\exp\left[\sum_{p=0}^{p_m+s-1} \delta f_{i,p}\theta_p(x,y)\right]. \tag{11}$$

Equation (11) is only correct for

$$\sum_{p=0}^{p_m+s-1} \delta f_{i,p}\theta_p(x,y) \ll 1, \qquad \text{in } F' \tag{12}$$

which means in the case of a small displacement of the eigenvalues stepping on from $i-1$ to i.

After the function $f_i(x,y)$ has been obtained, it is necessary to compute the corresponding eigenfunctions $\{\phi_{i,p}\}$ because they are needed for the computation of $f_{i+1}(x,y)$. That may be done by using variational techniques (method of Ritz) which means utilizing the stationary character of γ in (7) when $\phi(x,y)$ is an eigenfunction. Double Fourier series which satisfy the boundary conditions are introduced in (7). The derivations with respect to the unknown coefficients of the Fourier series lead to a matrix eigenvalue problem. A solution is found by the method of Wielandt [5] utilizing the fact that the eigenfunctions $\{\phi_{i-1,p}\}$ and eigenvalues $\{\gamma_{i-1,p}\}$ are good approximations.

To get the outline of the final longitudinal section F_{i_m}, only a solution of an ordinary first-order differential equation for a certain initial value must be found. A modified method of Euler–Cauchy is used [5].

A computer program (ALGOL) containing the formalism described in this section was written to obtain numerical results.

III. Numerical Results

An excitation of the nonuniform structure is assumed only as shown in Fig. 1(b). In order to save computer time and computer storage, only structures with a particular symmetry as shown in Fig. 2 were synthesized. In that case, the matrix equation of the eigenvalue problem is split up into four matrix equations (two of which are needed). The order of the matrices is thus reduced to a quarter of the original. The regions of integration in (7) and (8) are also reduced to a quarter. The conditions as in (10) may or may not be predefined.

The method was tested with a great number of examples. A great deal of experiences were gathered. The results were checked by methods given in literature or by actual measurements. The results were found to be in good agreement.

On account of the chosen series expansion for $f_i(x,y)$ it was not possible to influence the eigenvalues of the same order for the open- and the short-circuit case independently of one another. In this case, a singular matrix was obtained, when the coefficients $\delta f_{i,p}$ should be computed. In the same way, for some examples when both height B and length L of the longitudinal section F were preset, the method did not yield meaningful results. With length L as a free parameter, the process of synthesis was successful. There were no problems with a poor convergence of the eigenfunctions because the nonuniformity is always continuous. The nonuniformity is very critical if the longitudinal section F shows a tight pinch. In this case, the conformal mapping of the outline of the rectangular F' to the outline of the longitudinal section F automatically produces a great number of points in the critical region as shown in Fig. 4. On the other hand, it could also be observed that the conformal mapping was not unique for other examples of this type. The outline described a noose, which could be avoided by reducing the B/L ratio.

Now some characteristic examples are presented. In order to differentiate the eigenvalues in the short-circuit case and in the open-circuit case the letters K and L are used, respectively, instead of γ. Longitudinal sections shown are not in full scale.

At first, resonators were synthesized without thinking of their application. Examples where two, four, and six

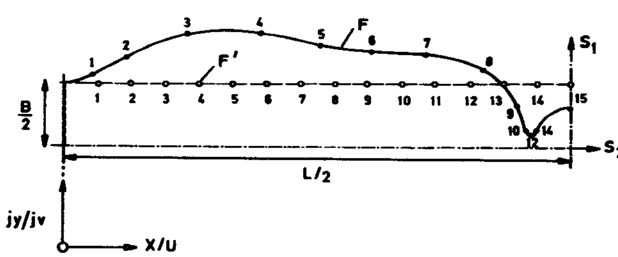

Fig. 4. Correlation of points on the outlines of F' and F by the conformal mapping.

eigenvalues of the short-circuit case were desired could be treated successfully.

Fig. 5 shows the step by step approximation of a particular longitudinal section F. In this case, certain values for only two eigenvalues $K1$ and $K2$ are desired. The initial values for $K1$ and $K2$ correspond to the TE_{101} and TE_{102} resonances, respectively, of the uniform waveguide resonator with the longitudinal section F'. For $K1$ and $K2$ a displacement of -45 and $+30$ percent, respectively, was required. In Fig. 6 the step by step approximation of the desired eigenvalues is shown. In Table I at first the desired incremental displacements in percent are given for each step and then the corresponding errors in percent when the desired eigenvalues are compared with the real (computed) eigenvalues of the new structure. These errors are always much smaller than the desired displacements. In this example, at first the eigenvalue $K1$ and then the eigenvalue $K2$ was displaced. This

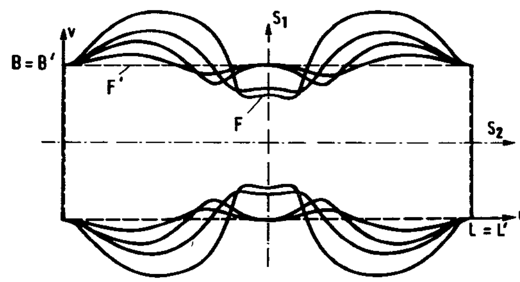

Fig. 5. Step by step approximation of a characteristic longitudinal section F.

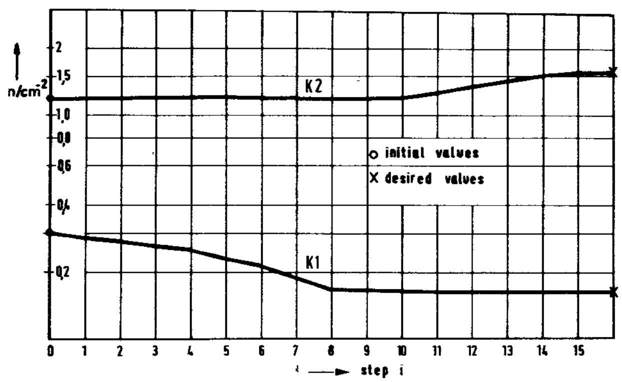

Fig. 6. Step by step approximation of the desired eigenvalues.

TABLE I
Accuracy of the Method during the Step by Step Approximation

	step i 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	desired displacement of the eigenvalues for step i−1→i in percent														
$K1/cm^{-2}$	−4	−2,6	−5,1	−4,5	−7,6	−7	−11,7	−10,1	−2,7	−0,3	−0,4	0	0	0	0
$K2/cm^{-2}$	0	0	0	0	0	0	0	0	0	0	6	7,3	6	5	2,5
	deviation of the real eigenvalues from the desired eigenvalues for each step in percent														
$K1/cm^{-2}$	0,15	−0,1	−0,1	−0,1	−0,35	−0,3	−1,2	−0,9	−0,32	0,05	0,03	−0,04	0,03	0,01	0,03
$K2/cm^{-2}$	0,46	0,03	0,19	0,07	0,16	0,03	−0,12	−0,13	−0,05	0,01	0,00	0,15	−0,18	−0,2	−0,09

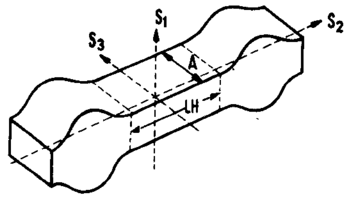

Fig. 7. Fundamental structure of a waveguide resonator for the synthesis of a reactive waveguide one-port.

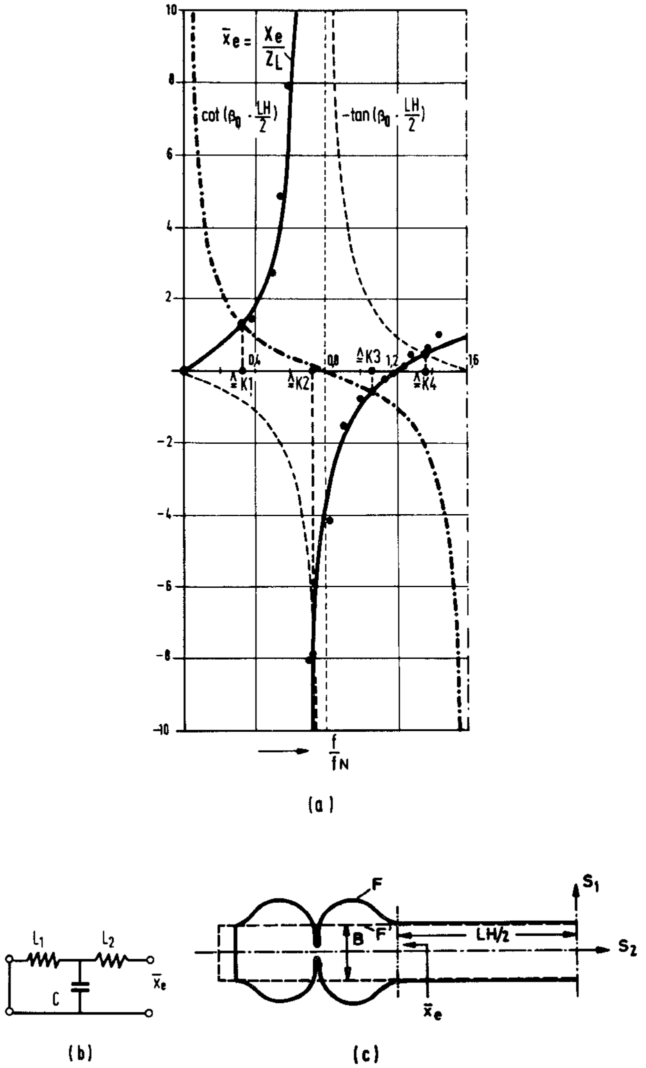

Fig. 8. Example for the synthesis of a reactive waveguide one-port. $f_N = 1.43$ GHz; $\beta_0 = 0.300 f/f_N$ cm^{-1}; $Z_L = B \cdot 120\pi\ \Omega$; $LH = 13.12$ cm; $B = 2.22$ cm.

is a method to avoid a singularity when a particular length L and height B is required.

Example two is the synthesis of a reactive waveguide one-port. In principle, a waveguide resonator as seen in Fig. 7 had to be synthesized. The identical nonuniform structures at both ends are the desired one-ports which are connected by a uniform waveguide (propagation constant β_0, characteristic impedance Z_L, width A, length LH) to obtain a resonator which will be computed as previously explained.

The solid curve in Fig. 8(a) shows the normalized input reactance $\bar{x}_e$ of a network corresponding to Fig. 8(b) to

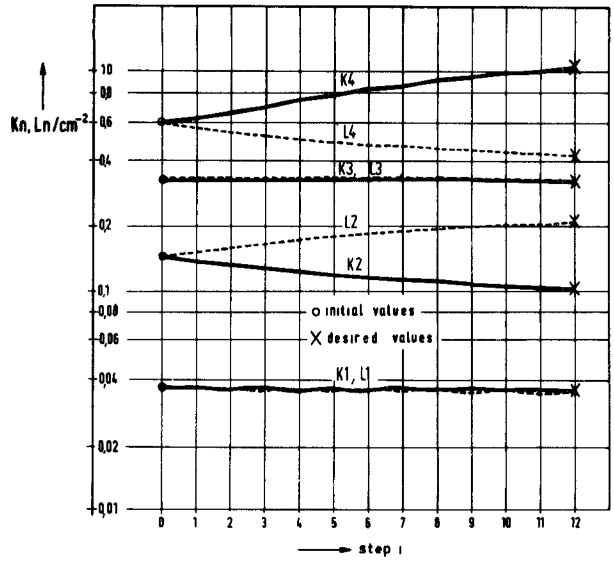

Fig. 9. Step by step approximation of the resonance spectrum of a bandpass filter.

be modeled in a waveguide one-port. For a given LH and A [compare (2)] taken to be infinite, the eigenvalues $K1 \cdots K4$ of the desired resonator are obtained by means of the dashed curves in Fig. 8(a).

The resonance conditions are

$$\bar{x}_e = \cot\left(\beta_0 \frac{LH}{2}\right) \quad \text{and} \quad \bar{x}_e = -\tan\left(\beta_0 \frac{LH}{2}\right). \tag{13}$$

In that way, the function $\bar{x}_e$ desired is obtained at discrete frequencies corresponding to the eigenvalues $K1 \cdots K4$. The density of the discrete frequencies depends on the length LH. Here, a series expansion somewhat modified as compared to (9) was used. A structure as shown in Fig. 8(c) was obtained. The electrical equivalent circuit of this structure agrees with the network in Fig. 8(b). A nonuniform waveguide one-port with the same longitudinal section was built and the input reactance $\bar{x}_e$ was measured. The relative agreement with the curve required is very good for lower frequencies, where the effect of higher modes (TE_{11}, TM_{11}) in the plane of definition of $\bar{x}_e$ can be neglected. The conversion of the results to the values shown in Fig. 8(a) was performed by a simple frequency transformation which can be derived from (2):

$$f_{A\to\infty}^2 = f_A^2 - \left(\frac{1}{2A}\right)^2 \cdot \frac{1}{\epsilon_0 \mu_0}. \tag{14}$$

Example three is the synthesis of a bandpass filter. In Fig. 9, the gradual approximation of the desired resonance spectrum is shown. At first, eigenvalues of the same order for the open- and short-circuit case agree. For special symmetrical structures, the following rule is valid [1]. Where the eigenvalues of the same order stay together ($K1$, $L1$ and $K3$, $L3$), a passband is obtained, where they diverge ($K2$, $L2$ and $K4$, $L4$), a stopband is obtained. The structure which belongs to the desired values can be seen in Fig. 10. It is comparable with a capacitive-iris

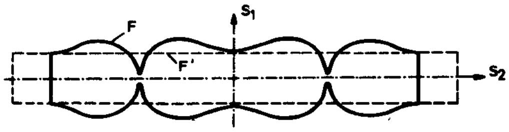

Fig. 10. Characteristic longitudinal section F of the bandpass filter developed.

coupled waveguide filter. At this example, it turned out that it was not possible to influence the eigenvalues of the same order for the open- and short-circuit case independently of one another. Therefore, they were influenced alternately.

The structure shown in Fig. 10 was analyzed by means of a stepped waveguide using formulas given in [6]. The location of the passbands was in very good agreement with the required values.

In the same way as just described, low passes and high passes were synthesized.

IV. Conclusion

A general principle for synthesizing nonuniform waveguides with desired properties was described. The method is an iterative one. The application of the method was described for one kind of nonuniform waveguide with rectangular cross section and excited by a TE_{10} mode. Simple examples have proved the feasibility of the method. Some experience for a successful adaptation of the method were given. In general, the method can be adapted to more complicated problems, e.g., matching problems. A corresponding computer program is under test.

Acknowledgment

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Short Papers

Tabulation of Methods for the Numerical Solution of the Hollow Waveguide Problem

FOOK LOY NG

***Abstract*—A comparison of methods for the numerical solution of the hollow waveguide problem is presented in tabular form. Another table lists waveguide shapes and their cutoff characteristics that have been presented in the literature. These tables and the bibliography afford an aid towards the selection of a method.**

Introduction

Consider a uniform waveguide with perfectly conducting walls. For the propagation of monochromatic electromagnetic waves inside the waveguide, Maxwell's equations reduce to the two-dimensional Helmholtz equation [1, sect. 8.1].

All analyses of the hollow waveguide problem are attempts at solving, exactly or approximately, the Helmholtz equation subject to the imposed Dirichlet or Neumann boundary conditions for E modes (TM) or H modes (TE), respectively [1, ch. 8].

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Many numerical methods have been proposed and used for the solution of the waveguide problem. A commentary on and comparison of the methods together with relevant references are given in this short paper. A table of the methods and their chief characteristics are presented for convenient reference. Another table is given listing the waveguide shapes that have been treated in the literature. This is provided as a handy reference of shapes that can be used for the testing of any numerical method. This short paper is a condensed version of an earlier publication appearing in a journal with limited circulation [2].

A general introduction to numerical techniques and a review of finite difference and variational techniques for electromagnetic problems are given by Wexler [3]. A review of some current numerical methods for the solution of the waveguide problem is given by Davies [4], and he establishes certain criteria as a basis for comparison of the various methods.

Comparison of Methods

Waveguide shapes can be classified [5] into the three basic types shown in Fig. 1.

In general, type 3 is the most troublesome computationally because of the singular behavior of the field at the reentrant corners [6, sect. 9.2]. Most of the methods either suffer from a slower convergence rate or do not produce reliable results for this type of shape.

The methods that have been used are compared in Table I. Some criteria established by Davies [4] for the comparison of